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TOPOLOGICAL CHARGE AND $U(1)_A$ SYMMETRY IN THE HIGH TEMPERATURE PHASE OF QCD

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Abstract

We discuss the global symmetries of the high temperature phase of QCD with N_f massless quarks. We show that the $U(N_f) \times U(N_f)$ symmetries are only violated by operators of dimension $\geq 3N_f$. For $N_f > 2$ this implies that the thermal two-point correlation functions of the η' and π^a's are identical. We discuss the implications of this for the chiral phase transition at finite temperature.

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Recently, Cohen [1] has used QCD inequalties to argue that the high temperature phase of massless QCD with N_f flavors of quarks is effectively symmetric under a global $U(N_f) \times U(N_f)$ symmetry rather than just $SU(N_f) \times SU(N_f)$. Cohen examines the two-point correlation functions of different operators such as the η' $(\bar{q}\gamma_5 q)$ and the π^a $(\bar{q}\tau^a\gamma_5 q)$ which are in the same $U(N_f) \times U(N_f)$ multiplet but *not* in the same $SU(N_f) \times SU(N_f)$ multiplet. He argues that in the massless (or chiral) limit: $m_q \to 0$, the difference between the respective two-point correlation functions approaches zero.

In this letter we show that the $U(N_f) \times U(N_f)$ symmetry is not completely restored in the high temperature phase although its breaking can only be manifested in operators of dimension $\geq 3N_f$. Thus the $U(N_f) \times U(N_f)$ symmetry is only restored for the two-point correlators when $N_f > 2$. We concentrate on the relationship between (spontaneous) chiral symmetry breaking, the axial anomaly and topological charge and clarify some subtle points in the argument of [1].

It is well-known that the solution of the $U(1)_A$ problem [2] requires

(A) the axial anomaly relation $(m_q = 0)$

$$\partial_\mu J^5_\mu \ = \ \tfrac{g^2 N_f}{16\pi^2} tr[F\tilde{F}], \tag{1}$$

and

(B) the presence of gauge configurations with non-zero topological charge

$$\nu \equiv \tfrac{1}{16\pi^2} \int d^4x \ tr[F\tilde{F}]. \tag{2}$$

We can understand this by examining the behavior of correlators under $U(1)_A$ transformations. Let[§]

$$\langle \mathcal{O} \rangle \ = \ \tfrac{1}{Z} \int D[A] \ e^{-S_{YM}} \ \int D\psi D\bar{\psi} e^{-\int \bar{\psi}(i\not{D} - m_q)\psi} \ \mathcal{O} \tag{3}$$

where $\mathcal{O}$ is an operator built out of $\psi, \bar{\psi}$ and gauge fields. Now consider $\mathcal{O}$ after an axial transformation:

$$\mathcal{O} \to \mathcal{O}_\alpha \ = \ \mathcal{O} \ (\bar{\psi}', \psi') \tag{4}$$

where

$$\psi \ \to \ \psi' \ = \ e^{i\alpha\gamma_5}\psi \tag{5}$$

$$\bar{\psi} \ \to \ \bar{\psi}' \ = \ \bar{\psi} e^{i\alpha\gamma_5}. \tag{6}$$

[§]We assume throughout this paper a non-perturbative regulator which conserves chiral symmetry and is controlled by a UV scale Λ. See, e.g., [3].

The change in the expectation of $\mathcal{O}$ is given by

$$\begin{aligned}\delta_\alpha \langle \mathcal{O} \rangle &= \langle \mathcal{O}_\alpha \rangle - \langle \mathcal{O} \rangle \\ &= \frac{1}{Z} \int D[A]\, e^{-S_{YM}} \int D\psi D\bar{\psi} e^{-\int \bar{\psi}(i\not{D} - m_q)\psi} \left[\mathcal{O}_\alpha - \mathcal{O} \right]. \end{aligned} \tag{7}$$

The integral in (7) can be evaluated by a change of variables: $\psi \to \psi'$, $\bar{\psi} \to \bar{\psi}'$. The only subtlety is that the Jacobian for the change of variables induces the anomaly factor in the measure of the functional integral [4]. In the chiral limit, this yields

$$\delta_\alpha \langle \mathcal{O} \rangle \;=\; \langle\, [e^{2i\alpha\nu} - 1]\; \mathcal{O} \rangle \;=\; \sum_{\nu \neq 0} \langle\, [e^{2i\alpha\nu} - 1]\; \mathcal{O} \rangle_\nu , \tag{8}$$

where $\langle \quad \rangle_\nu$ denotes the expectation value taken in the sector of the gauge field configuration space with topological charge ν. Thus we see that the physical effect of the $U(1)_A$ anomaly is only manifested in sectors of the functional integral with non-zero topological charge.

The operator relation in (1), which we used to derive (8) follows from the ultraviolet (UV) behavior of the theory and is therefore unaffected by temperature. One can easily see this by repeating the Fujikawa derivation [4] with boundary conditions in the Euclidean time direction which are appropriate for $T \neq 0$. As long as the UV regularization scale is kept large compared to T the same result for the anomalous variation of the functional measure is obtained. However, we will demonstrate below that above the temperature at which chiral symmetry is restored, the contributions of gauge configurations with $\nu \neq 0$ to correlators of quark operators of dimension $< 3N_f$ are suppressed (they are in fact a set of 'measure zero' in the functional measure) and hence the axial anomaly has no effect on the $U(1)_A$ Ward identities for these operators. For these operators we will show that the right hand side of (8) is zero when there is no spontaneous chiral symmetry breaking (i.e. in the high temperature phase of QCD) and the current quark masses m_q are taken to zero.

The presence of massless quarks is known to suppress topological fluctuations. The partition function for QCD is

$$\begin{aligned} Z &= \int D[A]\, e^{-S_{YM}}\; Det[\not{D} - m_q]\; e^{i\theta \int F\tilde{F}} \\ &\equiv \int [d\mu_A]\; e^{i\theta \int F\tilde{F}} \end{aligned} \tag{9}$$

where $Det[\not{D} - m_q] \;=\; \prod_n \; (i\lambda_n - m_q)$ and the λ_n are eigenvalues of the Euclidean Dirac equation: $\not{D}\psi_n = i\lambda_n \psi_n$. We can break Z into contributions from sectors of different winding number ν:

$$Z \;=\; \sum_{\nu=-\infty}^{\nu=+\infty} Z_\nu\; e^{i\theta\nu}. \tag{10}$$

An index theorem [2] tells us that there must exist a minimum number of zero mode solutions when $\nu \neq 0$: $\nu \;=\; n_+ - n_-$, where $n_\pm$ is the number of right(left)-handed solutions. This

implies that in the $\nu \neq 0$ sectors Z_ν must vanish at least as fast as $m_q^{|\nu| N_f}$ in the chiral limit. (In this paper we will assume exact $SU(N)_V$ isospin symmetry and take the quark mass matrix to be proportional to the identity matrix: $M = m_q \mathcal{I}$.) This seems to imply that only the $\nu = 0$ sector contributes to Z when massless quarks are present.

When chiral symmetry breaking is involved, this argument is too naive. This is fortunate since we believe that in QCD, which exhibits chiral symmetry breaking, the $U(1)_A$ problem is indeed solved by the combination of (A) and (B). Heuristically, one might guess that the 'dynamical mass' acquired by the quarks plays some role in determining whether topological fluctuations are suppressed. This issue has been addressed systematically by Leutwyler and Smilga [5]. They find that fluctuations in topological charge are controlled by the parameter $X = \Sigma\, V m_q$, where

$$\Sigma = \lim_{m \to 0} \lim_{V \to \infty} \tfrac{1}{V} \int d^4x \, \langle \bar{q} q(x) \rangle \tag{11}$$

is the chiral symmetry breaking order parameter, and V is the volume of the system. At large X topological fluctuations are allowed, whereas as $X \to 0$ they are suppressed. The order of limits in taking $V \to \infty$ and $m_q \to 0$ is important to the discussion here. If one takes $m_q \to 0$ with V fixed one does not recover the phase of QCD in which chiral symmetries are spontaneously broken. This is because symmetries cannot break spontaneously at finite volume, and hence for fixed V a non-zero m_q is required to bias the vacuum energy and keep the system in the broken vacuum. It is only at $X >> 1$ that finite-volume effects are small and we recover the low-energy vacuum of QCD. However in this limit topological fluctuations are unsuppressed even though m_q goes to zero. This is essentially a consequence of chiral symmetry breaking.

It is believed that in the high temperature phase of QCD chiral symmetry is restored by thermal effects. In other words, at sufficiently high temperature, the minimum of the free energy is at $\Sigma = 0$. We can study the theory at finite temperature by imposing (anti-) periodic boundary conditions in Euclidean time on the (fermion) boson fields appearing in the functional integral. If the period $\beta = 1/T$ is taken to be sufficiently small we will recover the high temperature phase. In this phase the subtlety associated with the order of limits $m_q \to 0$, $V \to \infty$ is no longer important[¶] as we are not trying to recover a phase with spontaneous symmetry breaking. Instead, we can take $m_q \to 0$ *before* taking the $V \to \infty$ (or the UV cutoff $\Lambda \to \infty$). This has important consequences, as it allows the naive scaling arguments in m_q to be used. It also allows us to make our arguments rigorous, as all quantities are finite while V and Λ are kept finite. For example, in the chirally restored phase Z_ν does indeed vanish like $m_q^{|\nu| N_f}$ in the chiral limit. In the absence of dynamical

[¶]For the order of limits to be important, some physical quantities would have to depend on parameters such as, e.g., $m_q L$, where L is the size of our box. However, this is highly implausible in the high-T (disordered) phase as there are neither long-range order nor long-range correlations in the heat bath.

chiral symmetry breaking massless quarks suppress the contributions to Z from sectors of non-trivial topology.

We next consider the contributions from sectors with $\nu \neq 0$ to arbitrary correlators in the high-temperature phase. By performing the fermionic part of the functional integral, the right hand side of (8) can be written as a sum over permutations of functional integrals over the gauge field measure with integrands that are products of traces of propagators, gamma matrices and flavor matrices (times possibly some functions of the gauge field, which we suppress):

$$\delta_\alpha \langle \mathcal{O} \rangle = \sum_{\nu \neq 0} \sum_{\text{perms}} \int [d\mu_A]\ _\nu\ [e^{i\alpha\nu} - 1]\ \prod\ tr \left[S_A^1(x_i, x_j)\Gamma_1 S_A^2(x_k, x_l)\Gamma_2 ... \right], \tag{12}$$

where Γ is any combination of gamma and flavor matrices.

In each ν sector the measure, $\int [d\mu_A]\ _\nu$, approaches zero as $m_q^{n_0}$ where n_0 is the number of zero modes and hence $n_0 \geq |\nu| N_f$. The propagators may be written in terms of their spectral decomposition

$$S_A(x,y) = \sum_k \frac{\psi_k^\dagger(x)\psi_k(y)}{\lambda_k - i m_q} \tag{13}$$

where it is important to remember that Fermi statistics (imposed by the integration over Grassmanian variables) forbids any two propagators in the product of sums from sharing an eigenvalue λ_k. Thus the integrands in (12) diverge at most as $(1/m_q)^n$ where the operator $\mathcal{O}$ is of dimension $3n$ but where the power of divergence must always be $\leq n_0$. The functional integral will never diverge as a result of the $m_q \to 0$ limit since the propagators in $\mathcal{O}$ may at most 'soak up' all n_0 zero modes ‖. However, we observe that correlators of all operators of dimension $< 3N_f$ receive no contributions from gauge configurations with $\nu \neq 0$ in the $m_q \to 0$ limit. The operators whose expectation values in $\nu \neq 0$ sectors are non-vanishing when $m_q \to 0$ are precisely the so-called 'tHooft operators [6] induced by instanton processes at weak coupling.

The two-point correlators are of special interest since they determine the number of (nearly) massless modes present at and above the high temperature phase transition, which is believed to be either second or weakly first order. For example, consider the two-point functions for the π and η' at finite temperature:

$$\langle \eta'(x)\eta'(0) \rangle \quad = \quad \langle\ \bar{\psi}_i \gamma_5 \psi_i\ (x)\ \bar{\psi}_j \gamma_5 \psi_j\ (0) \rangle \tag{14}$$

$$\langle \pi^a(x)\pi^a(0) \rangle \quad = \quad \langle\ \bar{\psi} \tau^a \gamma_5 \psi\ (x)\ \bar{\psi} \tau^a \gamma_5 \psi\ (0)\ \rangle \tag{15}$$

‖This is a loophole in the argument of [1] where it is argued that if the contribution to an an n-point correlator from a particular ν sector is non-zero the (n+2)-point quark correlator must diverge. If the n-point correlator is non-zero all zero modes in that sector have been soaked up by the propagators and the introduction of further propagators will not generate any further inverse powers of m_q. This loophole is problematic for the results of [1] when $N_f = 2$.

Writing these correlators in terms of exact quark propagators $S_A(x,y)$, one finds two types of contributions: a disconnected contribution

$$\tfrac{1}{Z}\int d\mu_A \; tr[\Gamma S_A(x,x)]Tr[\Gamma S_A(0,0)] \tag{16}$$

and a connected part

$$\tfrac{1}{Z}\int d\mu_A \; tr[S_A(x,0)\Gamma S_A(x,0)\Gamma]. \tag{17}$$

Here $\Gamma=\gamma_5$ for the η' and $\Gamma=\tau^a\gamma_5$ for the π^a. The connected parts (17) are identical since $[\tau^a, S_A]=0$. For the pion, the disconnected part is zero since $tr[\tau^a]=0$. Any η'-π^a splitting is the result of (16) for the η'.

The measure in a given ν sector goes to zero at least as fast as $m_q^{|\nu|N_f}$ as $m_q \to 0$ whilst the two propagators may soak up only two of the zero modes. Thus for $N_f>2$ the difference between the correlators vanishes as $m\to 0$ and the η' is degenerate with the massless pions. A similar result holds for the entire $U(N_f)\times U(N_f)$ multiplet which consists of the π,δ, σ and η' resonances.

When $N_f \leq 2$, however, there are potentially contributions to the correlator difference from the $\nu=\pm 1$ sector. There is good reason to believe that these contributions are non-zero since one-instanton effects which contribute to (16) were found in the weak coupling regime by 't Hooft [6]. His calculations in the weak coupling approximation are relevant at temperatures $T \gg \Lambda_{QCD}$, where the effective coupling constant is small and non-perturbative effects can be studied in the semiclassical approximation (see also [7]). When $N_f=2$ instanton effects contribute directly to (16), leading to an η'–π^a mass splitting. It is possible in principle that near the chiral phase transition $T\simeq T_c$, where the dilute instanton analysis is not completely reliable, that some other effects (such as I-$\bar{I}$ pair formation [8]) lead to the suppression of this splitting, but it is implausible that (16) vanishes completely.

In the case of $N_f=1$ there are potentially contributions to the chiral condensate

$$\Sigma = \frac{1}{Z}\int d\mu_A \; tr[S_A(x,x)] \tag{18}$$

from the $\nu=\pm 1$ sector. These can be seen to be non-zero even at high-temperature [7], and combining this with the low temperature analysis of Leutwyler and Smilga [5], one reaches the conclusion that the $U(1)_A$ violating chiral condensate stays non-zero for all temperatures with $U(1)_V$ being the only unbroken global symmetry.

We end with a summary of our conclusions for different values of N_f, and the corresponding implications for the chiral phase transition.

- $N_f=1$: There is no chiral phase transition, $\langle\bar{\psi}\psi\rangle$ remains non-zero at high temperatures, and the dynamically generated quark mass decreases smoothly to zero as $T\to\infty$.

- $N_f = 2$: The $SU(2)\times SU(2)$ global symmetry is restored in the high temperature phase. The η'–π^a splitting is non-zero, but decreases smoothly to zero with temperature as determined by the instanton density: ${m_{\eta'}}^2 \sim (\frac{\Lambda}{T})^k$. At $T \simeq T_c$ the η' is massive and the symmetry remains $SU(2) \times SU(2)$. Renormalization group calculations [9] (based on the ε-expansion) in an effective linear sigma model with this symmetry indicate that the phase transition is second order and QCD lattice simulations [10] appear to be in agreement with this prediction. It is worth mentioning that if the η'-π^a splitting is non-zero but *small* at $T \simeq T_c$, the effective model for the system as $T \to T_c$ from above may have an approximate $U(2) \times U(2)$ symmetry. Studies of $U(2) \times U(2)$ dynamics [9] generically lead to a fluctuation-induced first order transition, so it is possible that the chiral phase transition occurs through this instability before reaching the second order fixed point at length scales much larger than the η' correlation length. For an overview of these and related issues, see [8].

- $N_f \geq 3$ (assuming asymptotic freedom): The $U(N_f) \times U(N_f)$ global symmetry is effectively restored (up to high-dimension operators which are probably irrelevant in the IR limit) in the high temperature phase. The η' becomes degenerate with the π^a's for $T \geq T_c$. The effective models of this chiral phase transition should incorporate a $U(N_f) \times U(N_f)$ global symmetry. The analysis of the appropriate linear sigma model without determinantal interactions [9, 11] suggests that the phase transitions is first order with its strength increasing with N_f.

- Finally, it is amusing to note that in QCD theories with N_f just below $\frac{11}{2}N_c$, such that the theory has a perturbative infra-red fixed point, there is no dynamical chiral symmetry breaking and hence our results also apply there independent of the temperature. In such theories the $U(N_f) \times U(N_f)$ symmetry applies to all correlators up to some high dimension, where there are some exponentially small $U(1)_A$ violating interactions in the form of high-dimension instanton operators ('tHooft vertices). Strictly speaking there are no mesons here since we are at weak coupling.

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